Advanced Calculus Midterm Exam December 20, 2010

1. Let $\{a_k\}$ be a sequence defined recursively by $a_1 = \sqrt{3}$, and $a_{k+1} = \sqrt{3a_k}$, for $k = 1, 2, \ldots$.
   (a) (10 points) Show by induction that (i) $a_k < 3$ and (ii) $a_k < a_{k+1}$ for all k.
   (b) (10 points) Show that $\lim_{k\to\infty} a_k$ exists and evaluate it.

2. Let $S = \{(x,y) \in \mathbb{R}^2 \mid x, y \in \mathbb{Q}\}$.
   (a) (5 points) Find S', the set of all accumulation points of S. Justify your answer.
   (b) (5 points) Find ∂S, the set of all boundary points of S. Justify your answer.

3. (10 points) Show that the set $\{(x,y) \in \mathbb{R}^2 \mid f(x,y) = x - y + 1 = 0\}$ is closed in $\mathbb{R}^2$.
   [**Hint:** You may use the fact that $f(x,y) = x - y + 1$ is a continuous function on $\mathbb{R}^2$.]

4. (20 points) Let $S \subset \mathbb{R}^n$, $\mathbf{a} \in S$, and $f : S \to \mathbb{R}^n$. Show that the following are equivalent.
   (a) f is continuous at $\mathbf{a}$ in the sense of ε-δ definition, i.e. For each $\varepsilon > 0$ there exists a $\delta > 0$ such that if $|\mathbf{x} - \mathbf{a}| < \delta$ then $|f(\mathbf{x}) - f(\mathbf{a})| < \varepsilon$.
   (b) For any sequence of points $\{\mathbf{x}_k\}$ in S that converges to $\mathbf{a}$, the sequence $\{f(\mathbf{x}_k)\}$ converges to $f(\mathbf{a})$.
   [**Hint:** For (b) $\Rightarrow$ (a), you may want use a proof by contradiction.]

5. A set $S \subset \mathbb{R}^n$ is called pathwise connected if for any points x, y in S there is a continuous map $\mathbf{f} : [0,1] \to \mathbb{R}^n$ such that $\mathbf{f}(0) = x$, $\mathbf{f}(1) = y$, and $\mathbf{f}(t) \in S$ for all $t \in [0,1]$.
   (a) (10 points) Let $B(r, \mathbf{0}) = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| < r\}$ be the ball of radius r about the origin. Show that $B(r, \mathbf{0})$ is pathwise connected in $\mathbb{R}^n$..
   (b) (10 points) Show that if $S \subset \mathbb{R}^n$ is pathwise connected, then S is connected.

6. (10 points) Suppose that $\mathbf{f} : S \to \mathbb{R}^m$ and $\mathbf{g} : S \to \mathbb{R}^m$ are both uniformly continuous on S. Show that $\mathbf{f} + \mathbf{g}$ is uniformly continuous on S.

7. (10 points) Suppose $S \subset \mathbb{R}^n$ is compact, $f : S \to \mathbb{R}$ is continuous, and $f(x) > 0$ for every $x \in S$. Show that there is a number $c > 0$ such that $f(x) \geq c$ for every $x \in S$.